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Exercises, Examples, Insights

Exercise nr. 1

Show that for a single electron in Coulomb potential (hydrogen atom) the following relation holds:

$$\boldsymbol{\mu} = \boldsymbol{\mu}_{\ell} + \boldsymbol{\mu}_s = -\mu_B(\boldsymbol{\ell} + 2\mathbf{s}) = -g\mu_B\mathbf{j}$$

$$\mathbf{j} = \boldsymbol{\ell} + \mathbf{s}$$

In the hydrogen atom one expects to deal with the following four angular-momentum observables: $\boldsymbol{\ell}^2$, $\mathbf{s}^2$, ℓ_z , s_z .

However, the quantum vectors $\boldsymbol{\ell}$ e $\boldsymbol{s}$ are coupled by the spin-orbit interaction which can be cast in the form

$$\mathcal{H}_{SO} = \lambda\, \boldsymbol{\ell} \cdot \boldsymbol{s}$$

$$\lambda = \frac{e^2}{2m_e^2c^2\bar{r}^3}$$

Mean radius of the orbit

As a consequence, ℓ_z , s_z are no longer «good» quantum observables: the four diagonal operators become $\boldsymbol{\ell}^2$, $\mathbf{s}^2$, $\mathbf{j}^2$, $\mathbf{j}_z$, where $\boldsymbol{j}$ is the **total** angular momentum vector:

$$\boldsymbol{j} = (\boldsymbol{\ell} + \boldsymbol{s})$$

The eigenvalues of $\mathbf{j}^2$, $\mathbf{j}_z$ are $j(j+1)$ and $m_{j,z}$.

Both the magnitude of $\boldsymbol{j}$ and its projection along an arbitrary axis are «good quantum numbers»: the vector $\boldsymbol{j}$ of an unperturbed atom can be thought of as fixed in space, invariant.

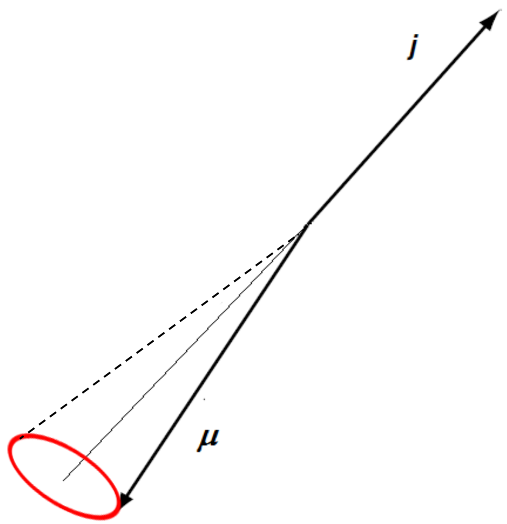
The magnetic moment μ is proportional to $\boldsymbol{\ell} + 2\mathbf{s}$, not to $\mathbf{j}$

However, its square modulus can be written as

$$\boldsymbol{\mu}^2 = \mu_B^2(\boldsymbol{\ell} + 2\mathbf{s})^2 = \mu_B^2\left(\boldsymbol{\ell}^2 + 4\mathbf{s}^2 + 4\boldsymbol{\ell}\cdot\mathbf{s}\right)$$

which is diagonal in the new basis ($\boldsymbol{\ell}\cdot\mathbf{s}$ is diagonal too).

We now show that it is possible to write: $\boldsymbol{\mu} = -g_j \mu_B \mathbf{j}$



where g_j (a dimensionless constant) is the *Landé's factor* generally taking the value:

$$g_j = \frac{3j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}$$

For a single electron s= ½ , so:

$$g_j = \frac{3j(j+1) + \frac{3}{4} - \ell(\ell+1)}{2j(j+1)}$$

We start by comparing the two expressions of the $\boldsymbol{\mu}$ operator:

$$\boldsymbol{\mu} = -g_j \mu_B \mathbf{j} \qquad \boldsymbol{\mu} = -\mu_B(\boldsymbol{\ell} + 2\mathbf{s}) \quad \Rightarrow \quad g_J \mathbf{j} = (\boldsymbol{\ell} + 2\mathbf{s})$$

to be shown

actual definition

The last expression transforms into:

$$g_J \mathbf{j} = (\boldsymbol{\ell} + 2\mathbf{s}) = \frac{3}{2}(\boldsymbol{\ell} + \mathbf{s}) - \frac{1}{2}(\boldsymbol{\ell} - \mathbf{s}) = \frac{3}{2}\mathbf{j} - \frac{1}{2}(\boldsymbol{\ell} - \mathbf{s})$$

Multiplying by **j** one easily finds:

$$g_J \mathbf{j}^2 = \frac{3}{2}\mathbf{j}^2 - \frac{1}{2}(\boldsymbol{\ell}^2 - \mathbf{s}^2)$$

which is again a diagonal operator with eigenvalues:

$$g_J j(j+1) = \frac{3}{2} j(j+1) - \frac{1}{2}\left(l(l+1) - s(s+1)\right)$$

Therefore:

$$g_J = \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$$

Exercise nr. 2

The Dy^{3+} ion has 9 electrons in the 4f shell (able to host up to 7+7= 14 electrons). Find the effective magnetic moment $\mu_{eff} = g_J\sqrt{J(J+1)}\mu_B$.

According to Hund's rules:

$$S = \frac{1}{2} + \frac{1}{2} + \cdots - \frac{1}{2} - \frac{1}{2} = \frac{7}{2} - \frac{2}{2} = \frac{5}{2}$$

$$L = 3 + 2 + 1 \ldots - 2 - 3 + 3 + 2 = 0 + 5 = 5$$

$$J = L + S = \frac{5}{2} + 5 = \frac{15}{2}$$

$$g_{\frac{15}{2}} = \frac{3\frac{15}{2}\left(\frac{15}{2}+1\right) + \frac{5}{2}\left(\frac{5}{2}+1\right) - 5(5+1)}{2\frac{15}{2}\left(\frac{15}{2}+1\right)} = \frac{\frac{45}{2}\frac{17}{2} + \frac{5}{2}\frac{7}{2} - 30}{\frac{30}{2}\frac{17}{2}} = \frac{765 + 35 - 120}{510} = \frac{680}{510} = \frac{4}{3}$$

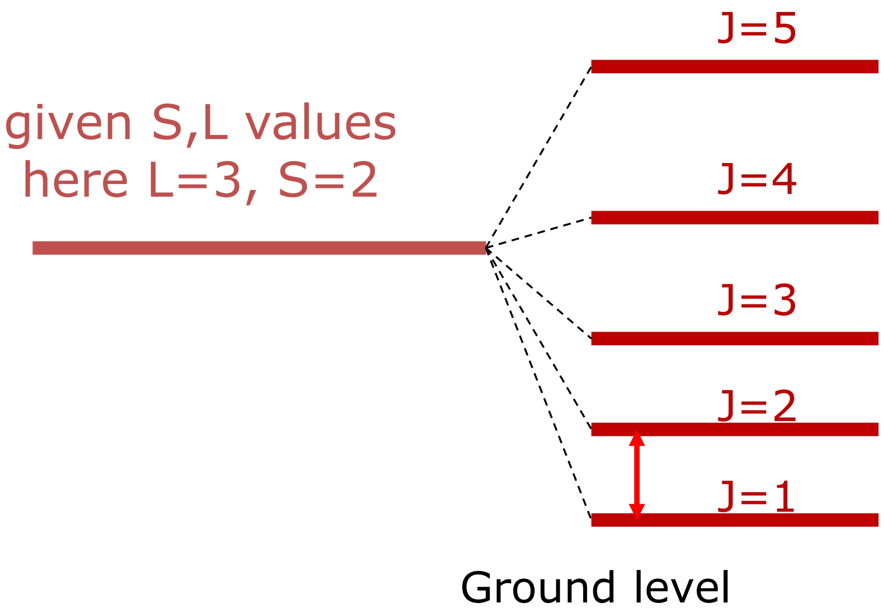
Therefore, $\boldsymbol{\mu_{eff}} = \frac{\mathbf{4}}{\mathbf{3}}\sqrt{\frac{\mathbf{15}}{\mathbf{2}}(\frac{\mathbf{15}}{\mathbf{2}} + \mathbf{1})}\boldsymbol{\mu_B} = \frac{\mathbf{4}}{\mathbf{3}}\sqrt{\frac{\mathbf{255}}{\mathbf{4}}} = \frac{\mathbf{2}}{\mathbf{3}} \times \mathbf{15.97}\boldsymbol{\mu_B} = \mathbf{10.65}\boldsymbol{\mu_B}$

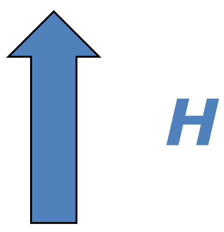
The experimental value of μ_{eff} for Dy^{3+} is 10.60 μ_B

Insight nr. 1

Multi-electron atoms/ions in the Russell-Saunders coupling scheme

An example & an important consequence



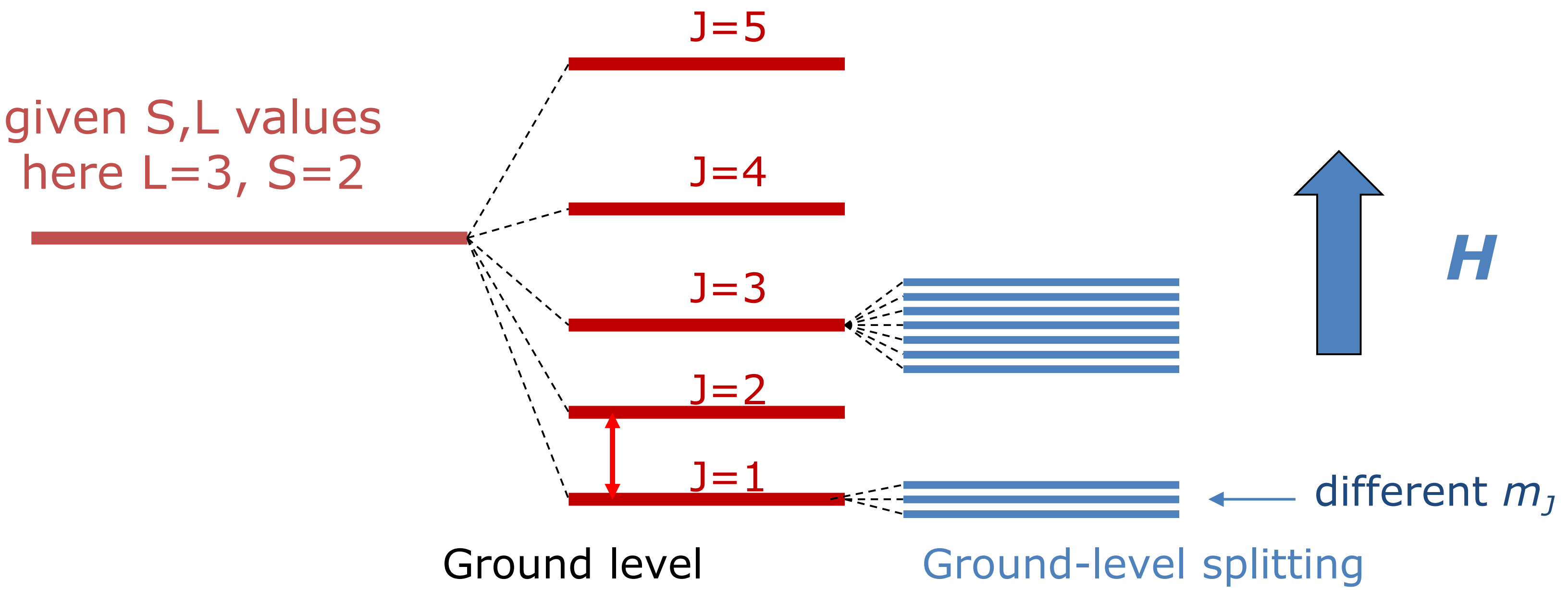


A magnetic field applied to the system removes the degeneracy of all levels with definite J because of the Zeeman interaction

↕ Energy difference between ground level and 1st excited level is proportional to spin-orbit interaction and is >> $k_B T$ around room temperature

Example & important consequence

Only ground level plays a role in magnetism at thermodynamic equlibrium



Energy difference between ground level and 1st excited level is proportional to spin-orbit interaction and is >> $k_B T$ at room temperature

Example & important consequence

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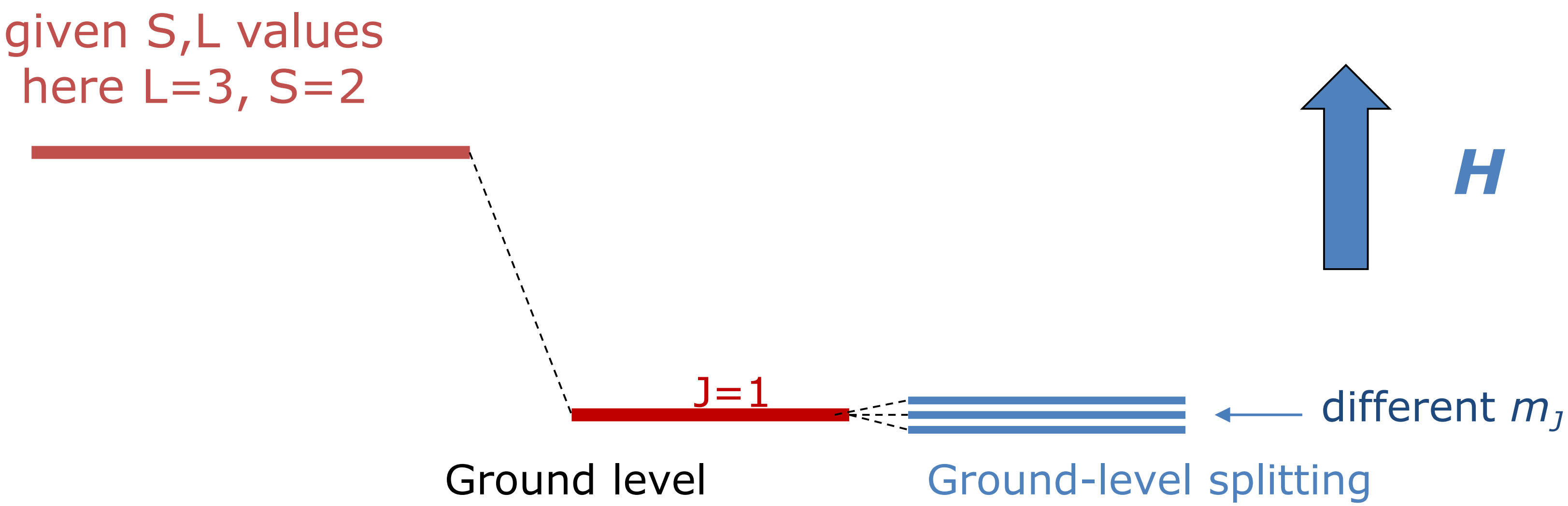


Energy difference between ground level and 1st excited level is proportional to spin-orbit interaction and is >> $k_B T$ at room temperature

Exercise nr. 3

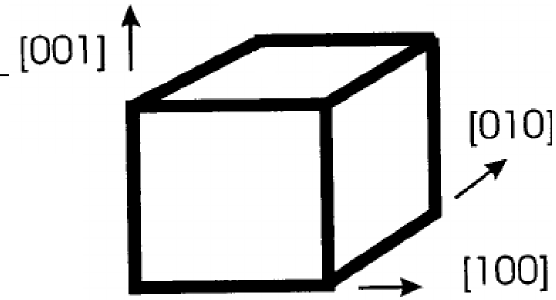
A paramagnetic system comprised of identical magnetic ions has a nonzero equlibrium magnetization M_0 when a large field H_0 is applied at RT. The field is instantaneously removed at time t=0.

- Which is the new equilibrium magnetization of the system?
- Which is the characteristic time needed by the system to reach the new equilibrium state?

- The new equilibrium magnetization is M=0
- The characteristic time is basically the spin-lattice relaxation time (10^{-9} to 10^{-4} s). Relaxation time decreases with increasing T.

Exercise nr. 4

Find the easy and hard anisotropy axes in cubic symmetry for $K_1>0$ and $K_1<0$. Neglect higher-order terms in the anisotropy development.



Cubic anisotropy: $E_A = K_0 + K_1(\alpha_1^2\alpha_2^2 + \alpha_2^2\alpha_3^2 + \alpha_3^2\alpha_1^2)$

$K_1 > 0$

Easy axes: Minimum of E_A (= K_0) when any $\alpha_i = 1$ (i=1-3) → the other two direction cosines must be zero → edge of cube

Hard axes: Maximum of E_A (= $K_0+K_1/3$) when $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{\sqrt{3}}$ → diagonal of the cube

$K_1 < 0$

Hard axes: Maximum of E_A (= K_0) when any $\alpha_i = 1$ (i=1-3) → the other two direction cosines must be zero → edge of cube

Easy axes: Minimum of E_A (= $K_0-K_1/3$) when $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{\sqrt{3}}$ → diagonal of the cube

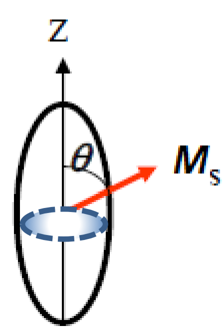
Exercise nr. 5

Show that M_s is always aligned along the long axis of an ellipsoid of revolution by shape anisotropy

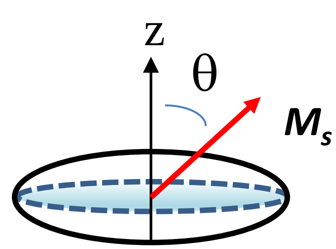
Evaluate shape anisotropy for important limiting cases

Shape anisotropy of the ellipsoid of revolution [$N_x = N_y = N_\perp \left(= 2\pi - \frac{N_z}{2}\right)$]

$$\varepsilon_d = const. + \frac{M_s^2}{8\pi}(N_z - N_\perp)cos^2\theta$$



Prolate ellipsoid (elongated along the z axis): $N_\perp > N_z$
→ minimum of ε_d for θ = 0, π (=z axis)



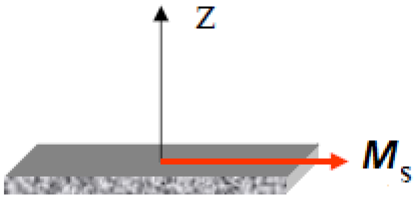
Oblate ellipsoid (a disk in the x-y plane): $N_\perp < N_z$
→ minimum of ε_d for $\theta = \pi/2$ (= in the plane of the disk)

Interesting limiting cases

Spherical sample: $N_x = N_y = N_\perp = N_z = \frac{4\pi}{3}$ → no shape anisotropy

Thin films: one dimension (z) is much smaller than the other two dimensions; according to the general rule $N_z >> N_x, N_y$, and $\rightarrow N_z \cong 4\pi$, $N_x \cong N_y \cong 0$.

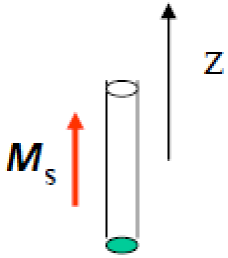
$$\varepsilon_d = const. + \frac{M_s^2}{8\pi} 4\pi cos^2\theta = const. + \frac{M_s^2}{2} cos^2\theta$$



The energy is minimized for $\theta = \pi/2$: the magnetization usually lies in the film's plane at equilibrium. A way to get a *perpendicular* equilibrium magnetization in a film is to use a material with an extremely high crystal anisotropy and easy axis along z.

Microwires: $N_z \cong 0$, $N_x \cong N_y \cong 2\pi$.

$$\varepsilon_d = const. - \frac{M_s^2}{4} cos^2\theta$$



The magnetization is spontaneously directed along the wire axis, θ=0,π (even if the wire is bent).

Insight nr. 2

The typical DW thickness *d* and the accumulated magnetic energy per unit of DW surface E_{DW} can be estimated making use of simple models.

E_{DW} is given in erg/cm^2 and is assimilated to a *surface tension*.

Example: Co

•d ≅ 24 nm

•E_{DW} ≅ 15 erg/cm^2

Example: Fe

•d ≅ 64 nm

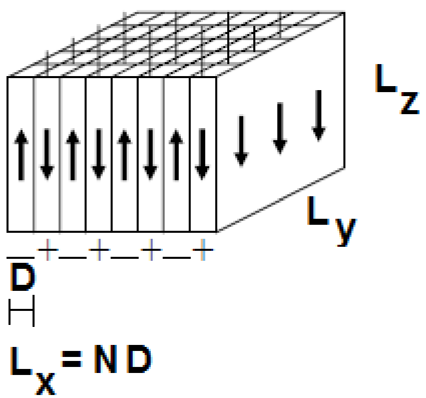
•E_{DW} ≅ 4 erg/cm^2

(surface tension of water at room temperature ≈ 70 erg/cm^2)

Insight nr. 3

Typical magnetic domain width *D* in Co, Fe according to simpl models.

DWs are genuine 2D structures if compared to the typical domain width.



N: number of DWs

Example: Co

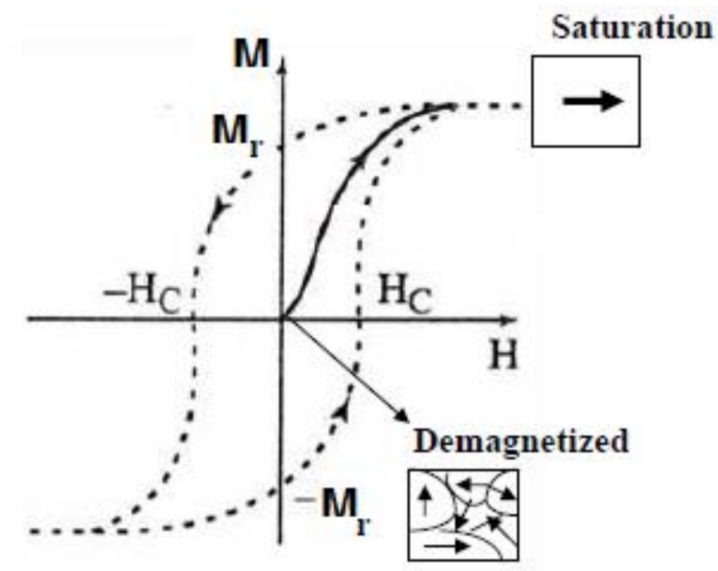
•D = 1 $\times 10^{-2}$ cm (100 μm)
•d = 24 nm

Example: Fe

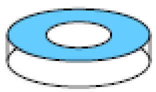
•D = 7 $\times 10^{-3}$ cm (70 μm)
•d = 64 nm

Exercise nr. 6

Show that the energy loss in cyclic magnetization of a ferromagnetic material is given by the hysteresis loop's area



Consider a toroidal core with area *A* and length *l.*



Energy loss over a period T:

$$E = \int_0^T i(t)V(t)dt$$

where i(t) is the eddy current flowing in the toroid and V is the electromotive force

The quantities in the integrand function are written as:

$$i(t) = \oint H(t)ds = H(t)l$$

Ampere's circuital law

$$V(t) = -\alpha \frac{dB}{dt} = -4\pi\alpha \frac{dM}{dt} \quad (\alpha = constant)$$

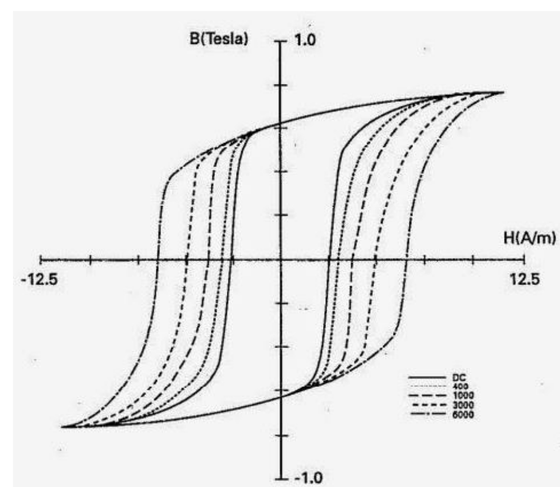
Faraday's law

so the energy loss is:

area of the M(H) loop

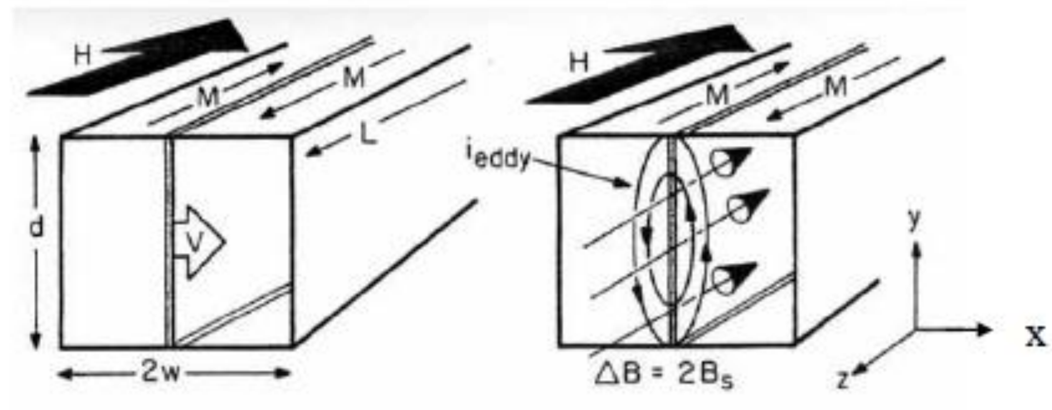
$$E = \int_0^T i(t)V(t)dt = \alpha l \int_o^T H(t)\frac{dB}{dt}dt = \alpha l \oint HdB = 4\pi\alpha l \oint HdM$$

When the loop is performed at a higher frequency, dM/dt and the electromotive force increase, so does the dissipated energy (the loop becomes wider).



Most of the energy loss depends on the **highly irregular motion** of DWs in the multi-valley energy potential landscape, occurring by a sequence of quick jumps forward followed by stasis.

This increases much the losses (dM/dt is very high during a single quick jump)



Note that the more conductive the material is, the higher its losses.

Suggested readings (all available online)

General Magnetism

J.M.D. Coey: *Magnetism and Magnetic Materials*, Cambridge University Press 2009
Accurate and updated; SI units
B.D. Cullity and C.D.Graham: *Introduction to Magnetic Materials*, Wiley 2009
Simple but exhaustive; gaussian units
J.B. Goodenough: *Magnetism and the chemical bond*, Wiley 1963
An old treatise; detailed depiction of quantum magnetism; gaussian units

Mostly Ferromagnetism

Soshin Chikazumi: *Physics of Ferromagnetism*, Oxford Science Publications 1997
In-depth analysis of phenomena in ferromagnetic materials; SI units

Magnetic Hysteresis

G. Bertotti: *Hysteresis in Magnetsim for Physicists*, Materials Scientists and Engineers, Academic Press 1998
Comprehensive exposition of stochastic magnetization processes; SI units

Magnetic Nanoparticles

Chris Binns (editor): *Nanomagnetism: Fundamentals and Applications*, Elsevier 2014
Title says it all; SI units