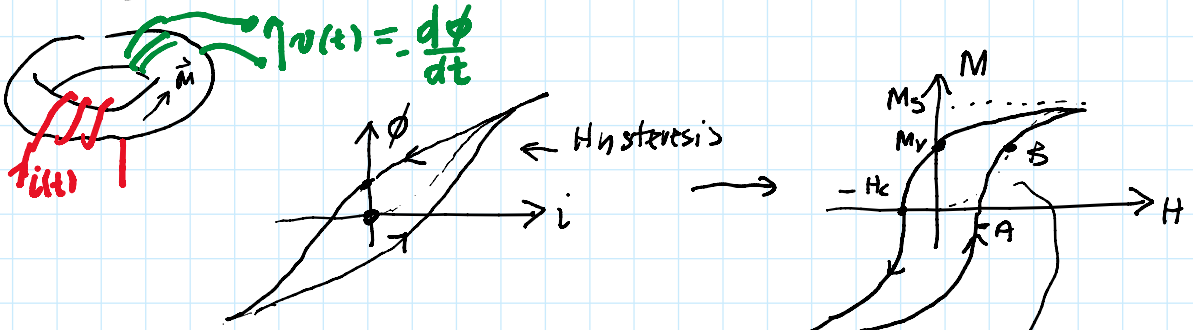
Ferromagnetism in micro- and nano-scale systems: Energy formulation

Outline

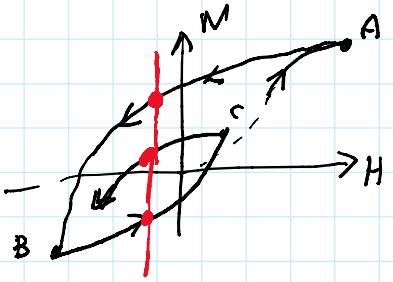
- Phenomenology of Ferromagnetism
- Magnetostatics
- Energy principle
- Micromagnetics

[BROWN63] Brown, Micromagnetics, Wiley (1963)
[BROWN62] Brown, Magnetostatic Principles in Ferromagnetism, NH (1962)
[Bertotti] Bertotti, Hysteresis in Magnetism, AP (1998)
[LL] Landau, Lifshitz, Electrodynamics of Continuous Media, Pergamon (1960)

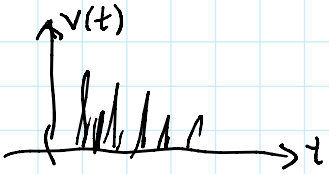
Phenom. of Ferromagnetism

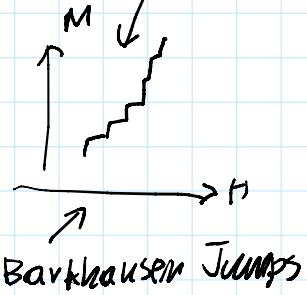


- Branching



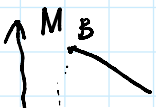
← Memory
rate-independent



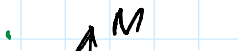


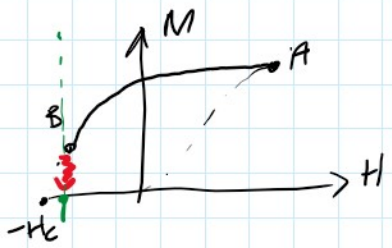
Barkhausen Jumps

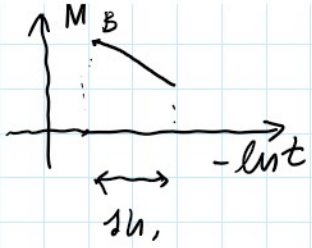
- Viscosity



- Multiplicity of metastable states

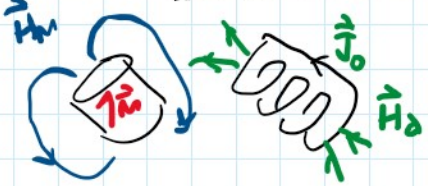






states

- Magnetostatic problem



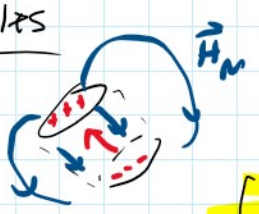
$$\begin{cases} \nabla\times\vec{H}_a = \vec{J}_0 \\ \nabla\cdot\mu_0\vec{H}_a = 0 \end{cases} \Rightarrow \vec{H}_a(\vec{x}) = \int \frac{\vec{J}_0(\vec{x}')\times(\vec{x}-\vec{x}')}{4\pi|\vec{x}-\vec{x}'|^3}\,dV \quad \text{Biot-Savart}$$

$$\begin{cases} \nabla\times\vec{H}_M = 0 \\ \nabla\cdot\vec{H}_M = -\nabla\cdot\vec{M} \end{cases} \overset{(1)}{\Longleftrightarrow} \nabla\cdot\vec{B}_M = 0 \qquad \vec{B}_M = \mu_0(\vec{H}_M+\vec{M})$$

$$\vec{H} = \vec{H}_a + \vec{H}_M = \vec{\mathcal{H}}(\vec{M}) \quad (3)$$

$$\vec{\mathcal{H}}(\vec{M}) = \frac{\vec{M}}{\chi_m(T)} \quad \text{linear}$$

- Magnetostatic principles



$$\begin{cases} \nabla\times\vec{H}_M = 0 \\ \nabla\cdot\vec{H}_M = -\nabla\cdot\vec{M} = \rho_M \\ \hat{n}\cdot(\vec{H}_M^{+}-\vec{H}_M^{-}) = \hat{n}\cdot\vec{M} = \sigma_M \end{cases}$$

$$\int_{V_M}\vec{M}\cdot\vec{H}_M\,dV \leqslant 0$$

- Reciprocity

$$\int\vec{M}_1\cdot\vec{H}_M[\vec{M}_2]\,dV = \int\vec{M}_2\cdot\vec{H}_M[\vec{M}_1]\,dV$$

$\Rightarrow \vec{H}_M$ is L^2 self-adjoint

- Tellegen theorem

$$\vec{v} = -\nabla\varphi\,; \quad \vec{w} = \nabla\times\vec{a}$$

$$\int_{V_\infty}\vec{v}\cdot\vec{w}\,dV = 0$$

- Helmholtz decomp.

$$\vec{v} = \vec{v}_c + \vec{v}_s \leftarrow \text{unique}$$

$\vec{v}_c$: cons. $\quad$ $\vec{v}_s$: solenoidal



$$\vec{B}_M = \mu_0(\vec{H}_M+\vec{M})$$

$$\vec{M} = \frac{\vec{B}_M}{\mu_0} - \vec{H}_M$$

$\vec{H}_M[\vec{M}]$, $\vec{B}_M[\vec{M}]$

↑ Helmholtz decomp. of $\vec{M}$

- Proof of reciprocity

$$\int_{V_\infty}\vec{M}_1\cdot\vec{H}_M[\vec{M}_2]\,dV = \int_{V_\infty}\left(\frac{\vec{B}_M[\vec{M}_1]}{\mu_0}-\vec{H}_M[\vec{M}_1]\right)\cdot\vec{H}_M[\vec{M}_2]\,dV =$$

$$= -\int_{V_\infty}\vec{H}_M[\vec{M}_1]\cdot\vec{H}_M[\vec{M}_2]\,dV$$

V_∞

- Positivity of $-\vec{H}_M[\vec{M}]$

$$\int_{V_\infty} -\vec{M}\cdot\vec{H}_M\,dV = \int_{V_\infty} -\left(\frac{\vec{B}_M}{\mu_0} - \vec{H}_M\right)\cdot\vec{H}_M\,dV = \int \vec{H}_M^2\,dV \geq 0$$

$$\vec{H}_M \equiv 0 \Longleftarrow \begin{cases} \nabla\cdot\vec{M} = 0 & \text{in } V_M \\ \vec{n}\cdot\vec{M} = 0 & \text{on } \partial V_M \end{cases} \Longrightarrow \int_{V_M} \vec{M}\,dV = 0$$

↑ Demagnetized state

$$\begin{cases} \nabla\times\vec{H}_M = 0 \\ \nabla\cdot\vec{H}_M = -\nabla\cdot\vec{M} \\ \vec{H}_0 + \vec{H}_M = \vec{\mathcal{H}}[\vec{M}] \end{cases}$$

- Monotonicity

$$\left[\vec{\mathcal{H}}(\vec{M}_1) - \vec{\mathcal{H}}(\vec{M}_2)\right]\cdot(\vec{M}_1 - \vec{M}_2) > 0 \qquad \vec{M}_1 \neq \vec{M}_2$$

Ex. $\vec{\mathcal{H}}(\vec{M}) = \frac{\vec{M}}{\chi_M}$ $\quad \chi_M > 0$

Proof uniqueness

$\vec{M}', \vec{M}''$ two solutions → $\vec{H}_M' = \vec{H}_M[\vec{M}']$; $\vec{H}_M'' = \vec{H}_M[\vec{M}'']$

$\delta\vec{M} = \vec{M}' - \vec{M}''$; $\delta\vec{H}_M = \vec{H}_M[\vec{M}' - \vec{M}'']$

$\delta\vec{\mathcal{H}} = \vec{\mathcal{H}}(\vec{M}') - \vec{\mathcal{H}}(\vec{M}'')$

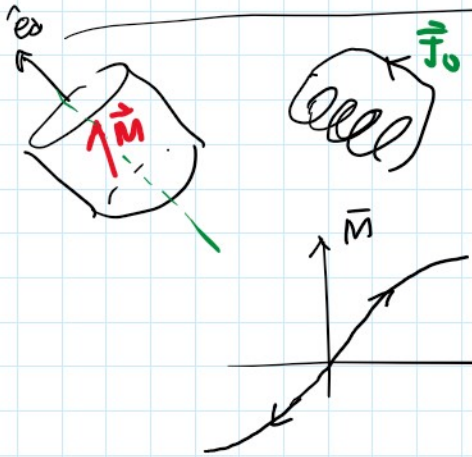
$\nabla\times\delta\vec{H}_M = 0$

$\nabla\cdot\delta\vec{H}_M = -\nabla\cdot\delta\vec{M}$

$\delta\vec{H}_M = \delta\vec{\mathcal{H}}$ ⇝

$$\int \delta\vec{M}\cdot\vec{H}_M[\delta\vec{M}]\,dV = \int \underbrace{\partial\vec{M}\cdot\delta\vec{\mathcal{H}}}_{\geq 0}\,dV \geq 0$$

$\vec{M}' = \vec{M}''$



$$\bar{M} = \frac{1}{V_M}\int_{V_M} \vec{M}\cdot\hat{e}_a\,dV$$

$$\bar{H}_a = \frac{1}{V_M}\int_{V_M} \vec{H}_a\cdot\hat{e}_a\,dV$$

← Takes care saturation, no Hysteresis

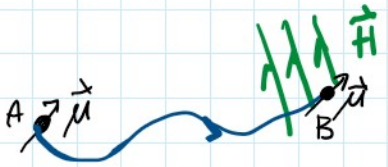
- Energy form. I

$\vec{H}$

$$\vec{F}_{ext} = -\vec{F}_\mu = -\mu_0\nabla(\vec{m}\cdot\vec{H})$$

$$F_{ext} = -F_\mu = -\mu_0 \nabla(\vec{\mu}\cdot\vec{H})$$

$$W_{\vec{\mu}} = -\mu_0 \vec{\mu}\cdot\vec{H}$$



$$\hookrightarrow W_{\vec{\mu}} = -\mu_0 \vec{\mu}\cdot\vec{H}_0$$



$$\delta\vec{\mu} = \delta\vec{M}\Delta V$$

- Postulate

$$\delta W_M = -\mu_0 \delta\vec{M}\Delta V \cdot \vec{H}_M[\vec{M}]$$

$$\delta W_M = -\mu_0 \int_{V_M} \delta\vec{M}\cdot\vec{H}_M[\vec{M}]\,dV$$

$W_M = ?$

- $W_M = -\int \frac{\mu_0}{2}\vec{M}\cdot\vec{H}_M[\vec{M}]\,dV$

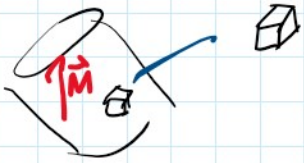
$$\delta\vec{H}_M[\vec{M}] = \vec{H}_M[\delta\vec{M}]$$

$$\delta W_M = -\int \frac{\mu_0}{2}\left\{\delta\vec{M}\cdot\vec{H}_M[\vec{M}] + \vec{M}\cdot\vec{H}_M[\delta\vec{M}]\right\}dV = -\int_{V_M} \mu_0\,\delta\vec{M}\cdot\vec{H}_M[\vec{M}]\,dV$$

$$W_M[\vec{M}] = \int_{V_M} -\frac{\mu_0}{2}\vec{M}\cdot\vec{H}_M[\vec{M}]\,dV$$

CVD

- Energy form. II . Local Thermodynamics



Thermodynamics

$$\begin{cases} du = dl + dq \\ ds = \frac{dq}{T} + \sigma \end{cases}$$

← per unit volume

$u \equiv$ internal energy

$\sigma > 0$, prod. of entropy

$dl = dl_{em}$

Poynting Theorem

$$-\nabla\cdot(\vec{E}\times\vec{H}) = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu_0\vec{H}^2 + \frac{1}{2}\varepsilon_0\vec{E}^2\right) + \vec{E}\cdot\vec{J}_0 + \vec{E}\cdot\frac{\partial\vec{P}}{\partial t} + \mu_0\vec{H}\cdot\frac{\partial\vec{M}}{\partial t}$$

Field - matter transfer of energy

$$dl_{em} = \mu_0\vec{H}\cdot\partial\vec{M}$$

$$du = \mu_0\vec{H}\cdot\delta\vec{M} + T\,ds - T\sigma \qquad \sigma = 0 \Rightarrow u = u(s,\vec{M})$$

$du = \mu_0 \vec{H} \cdot \delta\vec{M} + T\,dS - T\sigma \quad \Rightarrow \quad u = u(s, \vec{M})$

Helmholtz Free energy $\quad a = \min_s [u - Ts] = a(T, \vec{M})$

$$da = \mu_0 \vec{H} \cdot \delta\vec{M} - s\,dT - T\sigma$$

$\int dV$

$$A(T, \vec{M}(\cdot)) = \int_{V_M} a(T, \vec{M})\, dV$$

$$\delta A = \mu_0 \int_{V_M} (\vec{H}_a + \vec{H}_M[\vec{M}]) \cdot \delta\vec{M} - s\,dT - T\sigma$$

$$\delta\left[A - \int_{V_M} \mu_0 \vec{H}_a \cdot \delta\vec{M}\, dV\right] = \mu_0 \int_V \vec{H}_M \cdot \delta\vec{M}\, dV - s\,dT - T\sigma - \mu_0 \vec{M} \cdot \delta\vec{H}_a$$

V $\quad\quad = -\delta W_M$

$$\delta\left[A - \int_{V_M} \mu_0 \vec{H}_a \cdot \delta\vec{M}\, dV - \int_{V_M} \frac{\mu_0}{2} \vec{M} \cdot \vec{H}_M[\vec{M}]\, dV\right] = -\int_V s\,dT - \mu_0 \vec{M} \cdot \delta\vec{H}_a - T\sigma\, dV$$

$G_L[T, \vec{H}_a; \vec{M}]$

$$\Sigma = \int \sigma\, dV$$

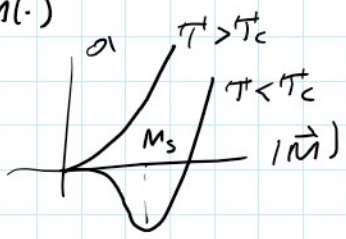
$$S = \int s\, dV$$

Trasf. with $dT = 0$; $\delta\vec{H}_a = 0$ $\quad\quad \nabla T = 0$

$$\delta G_L[T, \vec{H}_a; \vec{M}] = -T\Sigma \leq 0$$

Magnetization stable equilibria $\Longrightarrow$ $\min_{\vec{M}(\cdot)} G_L(T, \vec{H}_a; \vec{M})$



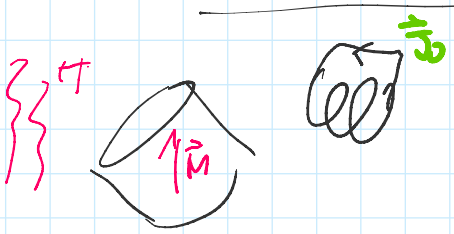


$$G_L[T, \vec{H}_a; \vec{M}] = \int_{V_M} \left[a(T, \vec{M}) - \frac{\mu_0}{2} \vec{M} \cdot \vec{H}_M[\vec{M}] - \mu_0 \vec{H}_a \cdot \vec{M}\right] dV$$

- Emerson principle $(\nabla T = 0)$ $\left(S = \int_V s\, dV\right)$ $\Sigma = \int_V \sigma\, dV$

• Energy principle $(\nabla T = 0)$ $\left(S = \int_{V_M} s\, dV \right)$ $\Sigma = \int_{V_M} \sigma\, dV$

$$\delta G_L = -S\,dT - \int_{V_M} \vec{m} \cdot \delta \vec{H}_a\, dV - T\Sigma \quad ; \quad \Sigma \geq 0$$



$\delta \vec{H}_a = 0$; $dT = 0$

$\delta G_L = -T\Sigma \leq 0$

$\Downarrow$

Stable equilibrium $\longrightarrow$ $\vec{M}(\cdot) = \underset{\vec{m}(\cdot)}{\arg\min}\ G_L(T, \vec{H}_a; \vec{M})$

Minima: $\begin{cases} \delta G_L = 0 \\ \delta^2 G_L \geq 0 \end{cases}$

$\vec{H}_M[\vec{M}]$ ≡ field gen. by mag. body

$$G_L = \int_{V_M} \left(a(T, \vec{M}) - \frac{\mu_0}{2} \vec{M} \cdot \vec{H}_M[\vec{M}] - \mu_0 \vec{H}_a \cdot \vec{M} \right) dV$$

$$\delta G_L = \int_{V_M} \left(\frac{\partial a}{\partial \vec{M}} - \mu_0 \vec{H}_M[\vec{M}] - \mu_0 \vec{H}_a \right) \cdot \delta \vec{M}\, dV = 0 \quad \forall\, \delta \vec{M}$$

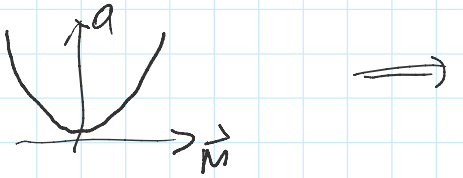
$$\vec{H}_M + \vec{H}_a = \frac{1}{\mu_0} \frac{\partial a}{\partial \vec{M}}\bigg|_T = \vec{H}(T, \vec{M})$$

$$\delta^2 G_L = \int_{V_M} \left(\frac{1}{2} \delta M^i \frac{\partial^2 a}{\partial M^i \partial M^j} \delta M^j - \frac{\mu_0}{2} \delta \vec{M} \cdot \vec{H}_M[\delta \vec{M}] \right) dV$$

≥ 0

$i, j = 1, 2, 3$ Einstein summation conv.

$\frac{\partial^2 a}{\partial M^i \partial M^j}$ pos. def $\Longrightarrow$ $\delta^2 G_L \geq 0$ $\Rightarrow$ $G_L[T, \vec{H}_a; \vec{M}]$ is a convex func.



Ex: $a(T, \vec{M}) - \mu_0 \vec{M}^2$; $\frac{\partial^2 a}{\partial \vec{M}^2}$ pos. def.

Ex: $a(T,\vec{M}) = \frac{\mu_0 \vec{M}^2}{\chi_m(T)}$;

$\frac{\partial^2 a}{\partial M^i \partial M^j}$ pos. def. $\Rightarrow \vec{\mathcal{H}} = \frac{\partial a}{\partial \vec{M}}$ is monotone

$[\vec{\mathcal{H}}(\vec{M}') - \vec{\mathcal{H}}(\vec{M}'')] \cdot (\vec{M}' - \vec{M}'') > 0$, $\vec{M}' \neq \vec{M}''$

$\nabla \times \vec{H}_M = 0$

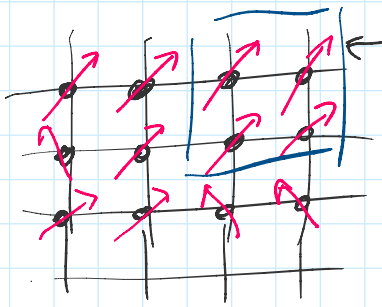
$\nabla \cdot \vec{H}_M = -\nabla \cdot \vec{M}$

$\vec{H}_a + \vec{H}_M = \vec{\mathcal{H}}(T,\vec{M}) = \frac{1}{\mu_0}\frac{\partial a}{\partial \vec{M}}$

$\Longleftrightarrow$ $\min_{\vec{M}(\cdot)} G_L(T, \vec{H}_0; \vec{M})$

→ $a(T,\vec{M})$ for ferromagnets ←

- Classical Heisenberg model (microscopic model)



$\vec{S}_i$

$|\vec{S}_i| = S$

$g_e \mu_B \vec{S}_i = \gamma \hbar \vec{S}_i = \vec{\mu}_i$

$H[\{\vec{S}_i\}] = \sum_{i,j=1}^{N} -J_{ij} \vec{S}_i \cdot \vec{S}_j +$ ← Exchange $J_{ij} > 0$

$+ \sum_{i=1}^{N} K_1 \left(1 - \frac{\vec{S}_i \cdot \vec{e}_{AN}}{S}\right) +$ ← Anisotropy

$\sum_{i,j}^{N} \frac{1}{2} \vec{S}_i \cdot \overleftrightarrow{N}_{ij} \cdot \vec{S}_j +$ ← Dipolar

$- \sum_{i=1}^{N} \mu_0 g_e \mu_B \vec{S}_i \cdot \vec{H}_{a,i}$

Coarse Graining

$F = \frac{1}{\Delta V} \int_{\Delta V} f \, dV$

$\int_{V_c} \vec{S}(\vec{x})\, dV = \vec{S}_i \rightarrow \vec{M}_\mu(\vec{x}) = -\gamma \hbar \vec{S}(\vec{x})$

↓ Average

$\vec{M}(\vec{x})$

Continuous Magnetization (→ Thermodynamics)

$M(\vec{x})$

- **Spontaneous Magnetization** ($\Leftrightarrow$ Ferromagnetism)

$$\mathcal{H}_{exc} = -\sum_{ij} J_{ij}\, \vec{S}_i \cdot \vec{S}_j$$

↓ Landau-Weiss Theory (mean field approx)

$$a(T, |\vec{M}|) = a_p(T, |\vec{M}|) - \frac{1}{2}\mu_0 N_w \vec{M}^2$$

← coming from exchange



$$a_p = \frac{\mu_0 |\vec{M}|^2}{2\chi_m(T)}$$



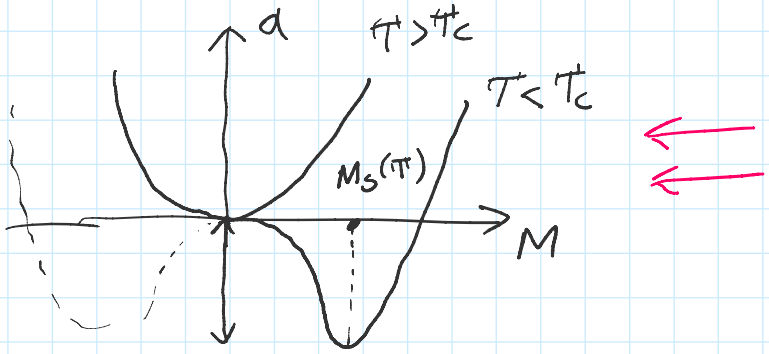
$$\chi_m = \frac{C}{T}$$

$$a_p = a_0 + \frac{\mu_0}{2}\frac{M^2}{\chi_m} + \beta M^4 - \frac{1}{2}\mu_0 N_w M^2$$

$$= a_0 + \frac{\mu_0}{2C} T M^2 - \frac{1}{2}\mu_0 N_w M^2 + \beta M^4$$

$$\frac{1}{2}\mu_0 N_w = \frac{\mu_0}{2C} T_c \Leftrightarrow T_c = C N_w$$

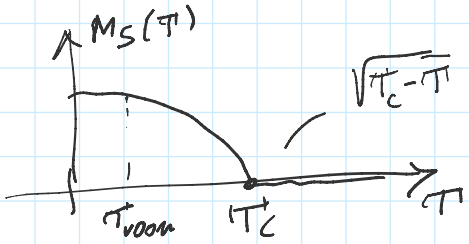
$$a = a_0 + \alpha (T - T_c) M^2 + \beta M^4$$



Micromagnetic constraint

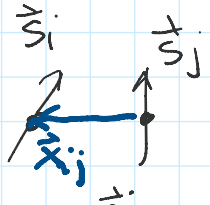
$$|\vec{M}| = M_S(T)$$

$$\vec{M}(\vec{x}, t) = M_S(T)\, \vec{m}(\vec{x}, t)$$



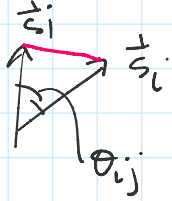
• Exchange additional terms

$$H_{exc} = -\sum_{i,j} J_{ij}\,\vec{s}_i\cdot\vec{s}_j \longrightarrow \begin{cases} |\vec{M}| = M_S(T) \\ a_{ex}(\nabla\vec{m}) = A(\nabla\vec{m})^2 \\ = A\left(|\nabla m_x|^2 + |\nabla m_y|^2 + |\nabla m_z|^2\right) \end{cases}$$



$$-J\vec{s}_i\cdot\vec{s}_j = -JS^2\cos(\theta_{ij})$$

$$-JS^2\left(1-\tfrac{1}{2}\theta_{ij}^2\right) \longrightarrow \tilde{A}\,\vec{x}_{ij}\left|\nabla\vec{s}(\vec{x})\right|^2$$



$$\theta_{ij} \approx \left|\frac{\vec{s}_i}{S} - \frac{\vec{s}_j}{S}\right| \simeq \left|\vec{x}_{ij}\,\nabla\vec{s}(\vec{x})\right|$$

• Micromagnetic free energy

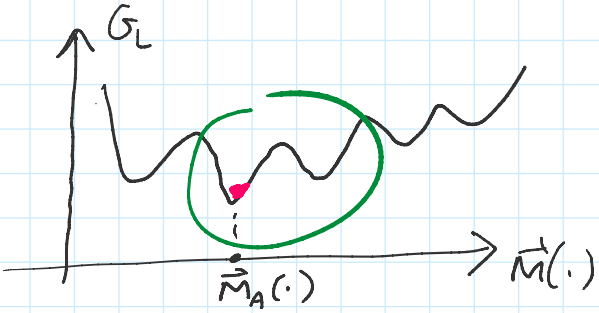
$$G_L(T, \vec{H}_a; \vec{M}) = \int_{V_M}\Big\{A(\nabla\vec{m})^2 + K_1\left[1-(\vec{m}\cdot\vec{e}_{AN})^2\right] + -\frac{\mu_0}{2}\vec{M}\cdot\vec{H}_M[\vec{M}] - \mu_0\vec{M}\cdot\vec{H}_a\Big\}\,dV$$

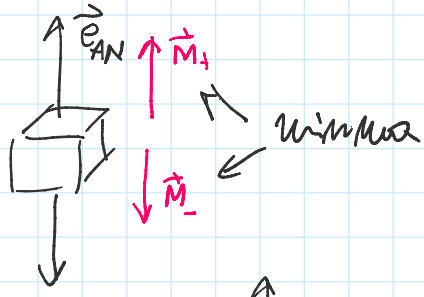
$$\begin{cases} |\vec{m}| = 1 \\ \vec{M} = M_S\vec{m} \end{cases}$$

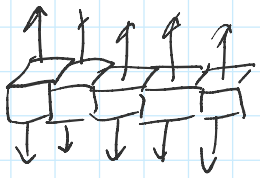


$M_S \simeq M_S(T)$

$A = A(T)$; $K_1 = K_1(T)$

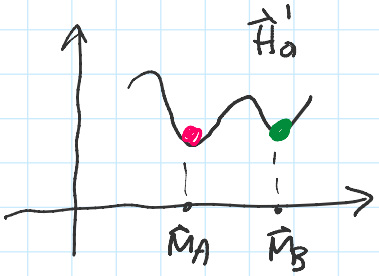




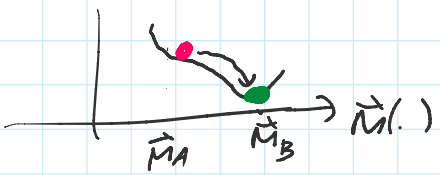


.... N cells

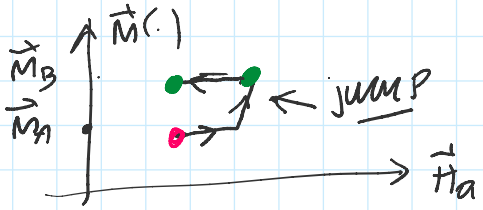
$\#\text{minima} \simeq 2^N$

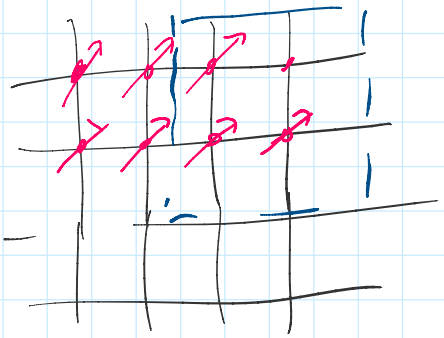


$\vec{H}_a'' > \vec{H}_a'$



$\vec{M}(\cdot)$

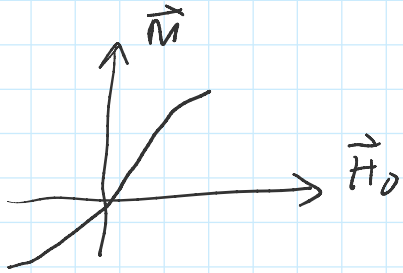




$\{\vec{S}_i\}$ ≡ state variable (micro)

$$\vec{M}(\{\vec{S}_i\},\vec{x}) = \frac{-\gamma\hbar \sum_{\Delta V(\vec{x})} \vec{S}_i}{\Delta V(\vec{x})}$$

$$\begin{cases} Z_{eq}(T,\vec{H}_a) = \sum_{\{\vec{S}_i\}} \exp\left[-H(\{\vec{S}_i\})/k_B T\right] \\ G(T,\vec{H}_a) = -k_B T \log Z(T,\vec{H}_a) \\ \vec{M}(T,\vec{H}_a) = -\frac{1}{\mu_0 V}\frac{\partial G}{\partial \vec{H}_a} \end{cases}$$

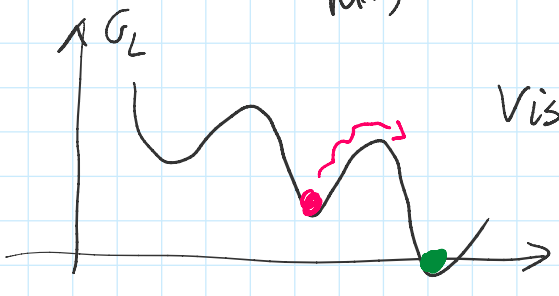


- Partial sum, constrained quasi-equilibria

$$Z_{eq}(T,\vec{H}_a) = \sum_{\vec{M}(\vec{x})} \sum_{\vec{M}(\{\vec{S}_i\},\vec{x})=\vec{M}(\vec{x})} \exp\left(-H(\{\vec{S}_i\})/k_B T\right)$$

$$\sum_{\vec{M}(\cdot)} \exp\left(-G_L(T,\vec{H}_a;\vec{M}(\cdot))/k_B T\right)$$

↖ effective Hamiltonian



Viscosity

$$\tau \sim \exp\left(\frac{\Delta G}{k_B T}\right)$$

$$G_L(T, \vec{H}_a; \vec{m}) = \int_{V_M} \left\{ A(\nabla \vec{m})^2 + K_1\left[1-(\vec{m}\cdot\vec{e}_{AN})^2\right] + \right.$$

$$\left. - \frac{\mu_0}{2}\vec{M}\cdot\vec{H}_M[\vec{M}] - \mu_0 \vec{M}\cdot\vec{H}_a \right\} dV$$

$\vec{M} = M_S \vec{m}$

- Equilibrio, Brown's eq.

$$\begin{cases} \delta G_L = 0 \\ |\vec{m}| = 1 \end{cases} \Rightarrow \int_{V_M} -\mu_0 \vec{H}_{eff}\cdot\delta\vec{M}\, dV = 0$$

$$\vec{H}_{eff} = A\nabla^2\vec{m} - \frac{1}{\mu_0}\frac{\partial a_{AN}}{\partial \vec{m}} + \vec{H}_M[\vec{M}] + \vec{H}_a$$

$|\delta\vec{M} + \vec{M}_0| = M_S$

↓ 1st order

$\vec{M}_0\cdot\delta\vec{M} = 0$

$\delta\vec{M} = \vec{M}_0 \times \delta\vec{\theta}$

$$\begin{cases} \vec{M}\times\vec{H}_{eff} = 0 & \text{in } V_M \\ |\vec{M}| = M_S \\ \hat{n}\cdot\nabla\vec{M} = 0 & \text{on } \partial V_M \end{cases}$$

Brown's Eqs.

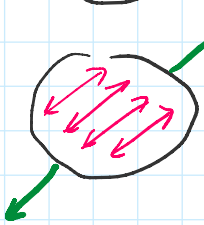
- Competing energies

Exchange $\int_{V_M} A(\nabla\vec{m})^2 dV \xrightarrow{min} \nabla\vec{m} = 0 \rightarrow$



← Spatially uniform

Anisotropy $\int_{V_M} K_1(1-(\vec{m}\cdot\vec{e}_{AN})^2)\, dV \rightarrow \vec{m} \parallel \vec{e}_A \rightarrow$



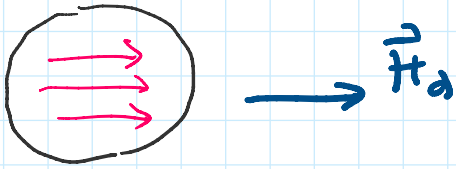
Dipolar $\int_{V_M} -\mu_0 \frac{\vec{M}\cdot\vec{H}_M}{2} dV \xrightarrow{min} \begin{cases} \nabla\cdot\vec{M} = 0 \text{ in } V \\ \vec{M}\cdot\hat{n} = 0 \text{ on } \partial V \end{cases}$



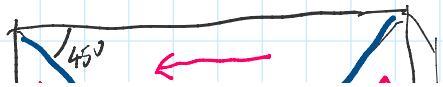
← Curling

Zeeman $\int_{V_M} -\mu_0 \vec{H}_a\cdot\vec{M}\, dV \xrightarrow{min} \vec{M} = c\vec{H}_a$

$c > 0$

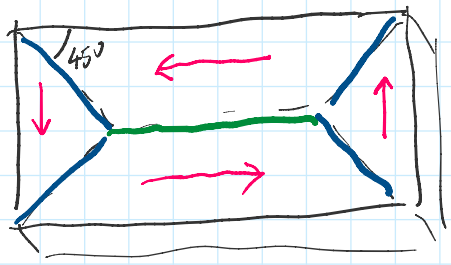


$\vec{H}_{a1} = 0$



$|\vec{M}| = M_S$

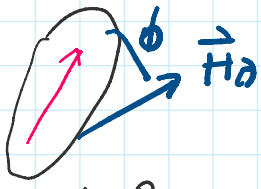
$\vec{H}_{ol} = 0$

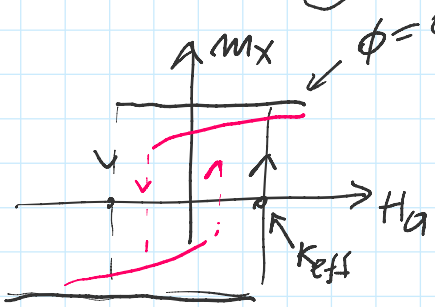


$|\vec{M}| = M_s$

$\vec{H}_M = 0$ $\uparrow \hat{e}_x$

$L \sim l_{ex}$





$$\int A (\nabla \vec{m})^2 dV$$

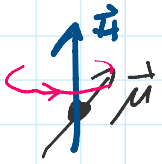
$$\mu_0 M_s^2 \int \frac{l_{ex}^2}{2} (\nabla \vec{m})^2 dV$$

$$l_{ex} = \sqrt{\frac{2A}{\mu_0 M_s^2}}$$

$l_{ex} \sim 10\ nm$

- **Dynamics**

- Larmor precession

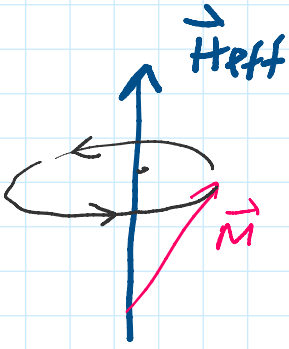


$$\frac{d\vec{L}}{dt} = -\mu_0 \vec{\mu} \times \vec{H} \equiv \text{Torque}$$

$$\vec{\mu} = \gamma \vec{L}$$

$$\frac{d\vec{\mu}}{dt} = -\mu_0 \gamma \vec{\mu} \times \vec{H}$$

$$\frac{\partial \vec{M}}{\partial t} = -\mu_0 \gamma \vec{M} \times \vec{H}_{eff}$$



$$\omega_L \sim \gamma(\vec{H}_{eff}) \sim 10\ GHz$$

$$\vec{H}_{eff} = -\frac{1}{\mu_0} \frac{\delta G_L}{\delta \vec{M}} \Longrightarrow \frac{\partial \vec{M}}{\partial t} = \mu_0 \gamma \vec{M} \times \frac{\delta G_L}{\delta \vec{M}}$$

$\Rightarrow$ $\int \left[\frac{\delta G}{\delta} \frac{\partial \vec{M}}{\partial} \right] dV = 0$

$$\frac{\delta G}{\partial \vec{M}} \cdot \frac{\partial \vec{M}}{\partial t} = 0 \qquad \frac{dG_L}{dt} = \int \frac{\delta G}{\delta \vec{M}} \cdot \frac{\partial \vec{M}}{\partial t} dV = 0$$

$\vec{H}_0 = \text{const}$

- Relaxation

$$\frac{\partial \vec{M}}{\partial t} = -\gamma \mu_0 \vec{M} \times \left(\vec{H}_{eff} - \alpha \frac{\partial \vec{M}}{\partial t} \right)$$

$\alpha \equiv$ Gilbert damping

$$= \gamma \mu_0 \vec{M} \times \left(\frac{\delta G}{\mu_0 \delta \vec{M}} + \alpha \frac{\partial \vec{M}}{\partial t} \right)$$

$$\frac{dG_L}{dt} = -\int_{V_M} \alpha \left| \frac{\partial \vec{M}}{\partial t} \right|^2 dV = -T\Sigma \qquad \frac{\partial \vec{H}_\alpha}{\partial t} = 0$$

